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ZALCMAN CONJECTURE FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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ABSTRACT. The aim of this investigation is to give a new subclass of analytic functions defined by Sălăgean differential operator and find upper bound of Zalcman functional $\left|a_n^2 - a_{2n-1}\right|$ for functions belonging to this subclass for $n = 3$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n\geq 2} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization conditions $f(0) = f'(0) - 1 = 0$.

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$ (for details, see [3]). We say that f is starlike on the open unit disk $\mathbb{U}$ with respect to origin, denoted by $f \in \mathcal{S}^\star$ if f is univalent on $\mathbb{U}$ and the image $f(\mathbb{U})$ is a starlike domain with respect to origin. Also, we say that f is convex on $\mathbb{U}$, denoted by $f \in \mathcal{C}$ if f is univalent on $\mathbb{U}$ and the image $f(\mathbb{U})$ is a convex domain in $\mathbb{C}$. A function $f \in \mathcal{S}$ is called starlike function of order α $(0 \leq \alpha < 1)$, denoted by $f \in \mathcal{S}^\star(\alpha)$, if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U}. \tag{1.2}$$

Moreover, we say that f is convex function of order α $(0 \leq \alpha < 1)$, denoted by $f \in \mathcal{C}(\alpha)$, if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in \mathbb{U}.$$

Nishiwaki and Owa [6] investigated the class $\mathcal{M}(\alpha)$ $(\alpha > 1)$ which is the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < \alpha, \quad z \in \mathbb{U} \tag{1.3}$$

and let $\mathcal{N}(\alpha)$ $(\alpha > 1)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha, \quad z \in \mathbb{U}. \tag{1.4}$$

Then, we observe that $f(z) \in \mathcal{N}(\alpha)$ if and only if $zf' \in \mathcal{M}(\alpha)$.

For convenience, we set $\mathcal{M}(3/2) = \mathcal{M}$ and $\mathcal{N}(3/2) = \mathcal{N}$. For $1 < \alpha \leq 4/3$, the classes of $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied Uralegaddi et al. [12]. Singh and Singh [11, Theorem

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6] proved that function in $\mathcal{N}$ are starlike in $\mathbb{U}$. Saitoh et al. [9] and Nunokawa [7] have improved the result of Singh and Singh [11, Theorem 6].

At the end of 1960's, Lawrence Zalcman posed a conjecture that the coefficients of $\mathcal{S}$ satisfy the sharp inequality

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2,$$

with equality only for the Koebe function and its rotations. This important conjecture implies the Bieberbach conjecture, scrutinized by many mathematicians, and still remains a very difficult open problem for all $n > 3$; it was proved only in certain special subclasses of S in [2, 5]. The case $n = 2$ is the elementary best-known Fekete-Szegö inequality. The more recently Bansal and Sokól [1] investigated the validity of Zalcman conjecture for $n = 3$ for the functions belonging to the classes $\mathcal{M}$ and $\mathcal{N}$ defined above.

For a function $f(z)$ belonging to $\mathcal{A}$, Sălăgean [10] has introduced the following differential operator called Sălăgean operator:

$$\begin{aligned} &D^0 f(z) = f(z); \\ &D^1 f(z) = Df(z) = zf'(z); \\ &\quad\vdots \\ &D^k f(z) = D(D^{k-1} f(z)) \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ where } \mathbb{N} = \{1, 2, 3, ...\}). \end{aligned}$$

We can easily observe that $D^k f(z) = z + \sum_{n \geq 2} n^k a_n z^n$.

Definition 1.1. *A function $f \in \mathcal{A}$ is said to be in the class $M_k(\alpha)$, if the following condition is satisfied:*

$$\text{Re}\left(\frac{D^{k+1} f(z)}{D^k f(z)}\right) < \alpha; \quad \alpha > 1, \ z \in U. \tag{1.5}$$

For convenience, we put $M_k(3/2) = M_k$. Taking $k = 0$ and $k = 1$ in Definition 1.1, we obtain that $M_0 \equiv \mathcal{M}$ and $M_1 \equiv \mathcal{N}$.

It is worth mentioning that the following lemma play a basic role in building our main result.

Lemma 1.1. *(see [8]) If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \quad (z \in \mathbb{U}), \tag{1.6}$$

then

$$|c_i| \leq 2 \quad \text{and} \quad |p_i - p_s p_{i-s}| \leq 2 \quad (i, s \in \mathbb{N}) \tag{1.7}$$

where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$ for which $p(0) = 1$ and $\text{Re}(p(z)) > 0$, $z \in \mathbb{U}$. Moreover, these inequalities are sharp for all i and for all s, equality being attained for each i and for each s by the function $p(z) = (1+z)/(1-z)$.

The second inequality in Lemma 1.1 was given by Livingston [4].

2. Main Results

Our main result is contained in the following theorem:

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class M_k. Then*

$$\left|a_3^2 - a_5\right| \leq \frac{1}{96.5^k 3^{2k}} \left(2\left|6.5^k - 3^{2k}\right| + \left|6.2.5^k - 10.3^{2k}\right| + 24.3^{2k}\right). \tag{2.1}$$

Proof. Let the function $f(z) \in M_k$ be given by (1.1), then there exists a function $p \in \mathcal{P}$ of the form (1.6), such that

$$\frac{D^{k+1}f(z)}{D^k f(z)} = \frac{1}{2}(3 - p(z)),$$

which in terms of power series is equivalent to

$$2D^{k+1}f(z) = \left(D^k f(z)\right)\left(2 - \sum_{n\geq 1} p_n z^n\right)$$

or

$$2\left(z + \sum_{n\geq 2} n^{k+1} a_n z^n\right) = \left(z + \sum_{n\geq 2} n^k a_n z^n\right)\left(2 - \sum_{n\geq 1} p_n z^n\right).$$

After some elementary calculations, we arrive at

$$a_2 = -\frac{1}{2.2^k} p_1, \tag{2.2}$$

$$a_3 = \frac{1}{8.3^k}\left(p_1^2 - 2p_2\right), \tag{2.3}$$

$$a_4 = \frac{1}{48.4^k}\left(6p_1p_2 - 8p_3 - p_1^3\right), \tag{2.4}$$

$$a_5 = \frac{1}{384.5^k}\left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right). \tag{2.5}$$

By using (2.3), (2.5) and Lemma 1.1, we arrive at

$$\begin{aligned}
\left|a_3^2 - a_5\right| &= \frac{1}{384}\left(\frac{6}{3^{2k}}\left(p_1^2 - 2p_2\right)^2 - \frac{1}{5^k}\left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right)\right) \\
&= \frac{1}{384.5^k.3^{2k}}\left(6.5^k\left(p_1^4 - 4p_1^2p_2 + 4p_2^2\right) - 3^{2k}\left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right)\right) \\
&= \frac{1}{384.5^k.3^{2k}}\Bigg(\left(6.5^k - 3^{2k}\right)\left(p_2 - p_1^2\right)^2 + \left(6.2.5^k - 10.3^{2k}\right)p_2\left(p_2 - p_1^2\right) \\
&\quad + \left(6.5^k - 3^{2k}\right)p_2^2 + 3^{2k}.32\left(p_4 - p_1p_3\right) + 3^{2k}p_4\Bigg) \\
&\leq \frac{1}{96.5^k.3^{2k}}\left(2\left|6.5^k - 3^{2k}\right| + \left|6.2.5^k - 10.3^{2k}\right| + 24.3^{2k}\right).
\end{aligned}$$

Thus, the proof of Theorem 2.1 is completed. □

Now, we would like to draw attention to some remarkable results which are obtained for some values of k in Theorem 2.1.

Taking $k = 0$ in Theorem 2.1 we obtain the following result.

Corollary 2.1 (see [1]). *Let the function $f \in \mathcal{M}$ be defined by (1.1), then*

$$\left|a_3^2 - a_5\right| \leq \frac{3}{8}.$$

The result is sharp.

Setting $k = 1$ in Theorem 2.1 we get the following result.

Corollary 2.2 (see [1]). *Let the function $f \in \mathcal{N}$ be defined by (1.1), then*

$$\left|a_3^2 - a_5\right| \leq \frac{1}{15}.$$

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