ZALCMAN CONJECTURE FOR SOME SUBCLASSES OF ANALYTIC FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. The aim of this investigation is to give a new subclass of analytic functions defined by Sălăgean differential operator and find upper bound of Zalcman functional \(|a_n^2 - a_{2n-1}|\) for functions belonging to this subclass for \(n = 3\).

1. Introduction

Let \(\mathcal{A}\) denote the class of functions \(f\) of the form
\[
f(z) = z + \sum_{n \geq 2} a_n z^n
\]
which are analytic in the open unit disk \(U := \{z \in \mathbb{C} : |z| < 1\}\) and satisfy the normalization conditions \(f(0) = f'(0) - 1 = 0\).

We also denote by \(\mathcal{S}\) the class of all functions in the normalized analytic function class \(\mathcal{A}\) which are univalent in \(U\) (for details, see [3]). We say that \(f\) is starlike on the open unit disk \(U\) with respect to origin, denoted by \(f \in \mathcal{S}^\ast\) if \(f\) is univalent on \(U\) and the image \(f(U)\) is a starlike domain with respect to origin. Also, we say that \(f\) is convex on \(U\), denoted by \(f \in \mathcal{C}\) if \(f\) is univalent on \(U\) and the image \(f(U)\) is a convex domain in \(\mathbb{C}\). A function \(f \in \mathcal{S}\) is called starlike function of order \(\alpha\) \((0 \leq \alpha < 1)\), denoted by \(f \in \mathcal{S}^\ast(\alpha)\), if
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U.
\]
Moreover, we say that \(f\) is convex function of order \(\alpha\) \((0 \leq \alpha < 1)\), denoted by \(f \in \mathcal{C}(\alpha)\), if
\[
\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U.
\]
Nishiwaki and Owa [6] investigated the class \(\mathcal{M}(\alpha)\) \((\alpha > 1)\) which is the subclass of \(\mathcal{A}\) consisting of functions \(f(z)\) which satisfy the inequality
\[
\Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha, \quad z \in U
\]
and let \(\mathcal{N}(\alpha)\) \((\alpha > 1)\) be the subclass of \(\mathcal{A}\) consisting of functions \(f(z)\) which satisfy the inequality
\[
\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha, \quad z \in U.
\]
Then, we observe that \(f(z) \in \mathcal{N}(\alpha)\) if and only if \(zf' \in \mathcal{M}(\alpha)\).

For convenience, we set \(\mathcal{M}(3/2) = \mathcal{M}\) and \(\mathcal{N}(3/2) = \mathcal{N}\). For \(1 < \alpha \leq 4/3\), the classes of \(\mathcal{M}(\alpha)\) and \(\mathcal{N}(\alpha)\) were studied Uralegaddi et al. [12]. Singh and Singh [11, Theorem

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proved that function in \( N \) are starlike in \( U \). Saitoh et al. [9] and Nunokawa [7] have improved the result of Singh and Singh [11, Theorem 6].

At the end of 1960’s, Lawrence Zalcman posed a conjecture that the coefficients of \( S \) satisfy the sharp inequality
\[
|a_{2n}^2 - a_{2n-1}| \leq (n - 1)^2,
\]
with equality only for the Koebe function and its rotations. This important conjecture implies the Bieberbach conjecture, scrutinized by many mathematicians, and still remains a very difficult open problem for all \( n > 3 \); it was proved only in certain special subclasses of \( S \) in [2, 5]. The case \( n = 2 \) is the elementary best-known Fekete-Szegő inequality. The more recently Bansal and Sokół [1] investigated the validity of Zalcman conjecture for \( n = 3 \) for the functions belonging to the classes \( M \) and \( N \) defined above.

For a function \( f(z) \) belonging to \( A \), Salagean [10] has introduced the following differential operator called Salagean operator:
\[
D^0 f(z) = f(z);
\]
\[
D^1 f(z) = D f(z) = zf'(z);
\]
\[
D^k f(z) = D(D^{k-1} f(z)) \quad (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \text{ where } \mathbb{N} = \{1, 2, 3, \ldots\}).
\]
We can easily observe that
\[
D^k f(z) = z + \sum_{n \geq 2} n^k a_n z^n.
\]

**Definition 1.1.** A function \( f \in A \) is said to be in the class \( M_k(\alpha) \), if the following condition is satisfied:
\[
\text{Re} \left( \frac{D^{k+1} f(z)}{D^k f(z)} \right) < \alpha; \quad \alpha > 1, \ z \in U.
\]

For convenience, we put \( M_k(3/2) = M_k \). Taking \( k = 0 \) and \( k = 1 \) in Definition 1.1, we obtain that \( M_0 \equiv M \) and \( M_1 \equiv N \).

It is worth mentioning that the following lemma play a basic role in building our main result.

**Lemma 1.1.** (see [3]) If a function \( p \in \mathcal{P} \) is given by
\[
p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \quad (z \in U),
\]
then
\[
|c_i| \leq 2 \quad \text{and} \quad |p_i - p_s p_{i-s}| \leq 2 \quad (i, s \in \mathbb{N})
\]
where \( \mathcal{P} \) is the family of all functions \( p, \) analytic in \( U \) for which \( p(0) = 1 \) and \( \text{Re}(p(z)) > 0, \ z \in U \). Moreover, these inequalities are sharp for all \( i \) and for all \( s \), equality being attained for each \( i \) and for each \( s \) by the function \( p(z) = (1 + z)/(1 - z) \).

The second inequality in Lemma 1.1 was given by Livingston [4].

2. Main Results

Our main result is contained in the following theorem:

**Theorem 2.1.** Let the function \( f(z) \) given by (1.1) be in the class \( M_k \). Then
\[
|a_3^2 - a_5| \leq \frac{1}{96.5^k 3^{2k}} \left( 2 |6.5^k - 3^{2k}| + |6.2.5^k - 10.3^{2k}| + 24.3^{2k} \right).
\]
Proof. Let the function \( f(z) \in M_k \) be given by (1.1), then there exists a function \( p \in P \) of the form (1.6), such that
\[
\frac{D^{k+1}f(z)}{D^kf(z)} = \frac{1}{2}(3 - p(z)),
\]
which in terms of power series is equivalent to
\[
2D^{k+1}f(z) = (D^kf(z)) \left(2 - \sum_{n \geq 1} p_n z^n\right)
\]
or
\[
2 \left(z + \sum_{n \geq 2} n^{k+1} a_n z^n\right) = \left(z + \sum_{n \geq 2} n^k a_n z^n\right) \left(2 - \sum_{n \geq 1} p_n z^n\right).
\]
After some elementary calculations, we arrive at
\[(2.2)\quad a_2 = -\frac{1}{2.5k} p_1, \]
\[(2.3)\quad a_3 = \frac{1}{8.3k} \left(p_1^2 - 2p_2\right), \]
\[(2.4)\quad a_4 = \frac{1}{48.4k} \left(6p_1p_2 - 8p_3 - p_1^3\right), \]
\[(2.5)\quad a_5 = \frac{1}{384.5k} \left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right).
\]
By using (2.3), (2.5) and Lemma 1.1, we arrive at
\[
|a_3^2 - a_5| = \frac{1}{384} \left(\frac{6}{32k} \left(p_1^2 - 2p_2\right)^2 - \frac{1}{5k} \left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right)\right)
\]
\[
= \frac{1}{384.5k.32k} \left(6.5^k \left(p_1^4 - 4p_1^2p_2 + 4p_2^2\right) - 3^2k \left(p_1^4 + 12p_2^2 + 32p_1p_3 - 48p_4 - 12p_1^2p_2\right)\right)
\]
\[
= \frac{1}{384.5k.32k} \left(\left(6.5^k - 3^2k\right) \left(p_2 - p_1^2\right)^2 + \left(6.2.5^k - 10.3^2k\right) p_2 \left(p_2 - p_1^2\right) + \left(6.5^k - 3^2k\right) p_2^2 + 3^2k \cdot 32 \left(p_4 - p_1p_3\right) + 3^2k \cdot 32 \right)
\]
\[
\leq \frac{1}{96.5^k.32k} \left(2 \left|6.5^k - 3^2k\right| + \left|6.2.5^k - 10.3^2k\right| + 24.3^2k\right).
\]
Thus, the proof of Theorem 2.1 is completed. \( \square \)

Now, we would like to draw attention to some remarkable results which are obtained for some values of \( k \) in Theorem 2.1.

Taking \( k = 0 \) in Theorem 2.1, we obtain the following result.

**Corollary 2.1 (see [1]).** Let the function \( f \in \mathcal{M} \) be defined by (1.4), then
\[
|a_3^2 - a_5| \leq \frac{3}{8}.
\]
The result is sharp.

Setting \( k = 1 \) in Theorem 2.1, we get the following result.

**Corollary 2.2 (see [1]).** Let the function \( f \in \mathcal{N} \) be defined by (1.4), then
\[
|a_3^2 - a_5| \leq \frac{1}{15}.
\]
References


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