Some Results on Especial Diophantine Sets with Size-3

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Abstract

Purpose of this paper is to determine some regular non-extendible $D(n)$ triples for some fixed integer $n$. Besides, paper includes a number of algebraic properties for such diophantine sets with size three.

Keywords: Diophantine Sets, Property $D(n)$, Integral Solutions of Pell Equations, Quadratic Residues and Reciprocity Theorems, Legendre Symbol.

2010 Mathematics Subject Classification: 11D09, 11A15, 11A07.

1. Introduction and Preliminaries

There are a lot of significant and attracted results in the literature related with Diophantine sets and equations. One of them was started by a Greek mathematician Diophantus of Alexandria in the 3rd century. Mathematicians have been interested in Diophantine sets for a long time before due to unsolved problems in the literature.
For this paper, we use some basic notions such as quadratic reciprocity and residues ([3, 7, 13-16, 22]), Legendre symbol ([5, 12]) and Diophantine sets with their regularity ([6,8-10, 17-21]) as well as significant books ([1, 2, 4, 6, 10, 16]) from algebraic and elementary number theories. We obtain regular non-extendibility of some $D(n)$ Diophantine triples where $n$ is 31 or -31. Additionally, we demonstrate that some types of elements cannot be in $D(\mp 31)$.

**Definition 1.1.** ([6,8,9]) A Diophantine $m$-tuple with the property $D(n)$ (it sometimes represents as $P_n$ with $m$-tuples) for $n$ an integer is an $m$-tuple of different positive integers $\{\beta_1, \ldots, \beta_n\}$ such that $\beta_i \beta_j + n$ is always a square of an integer for every distinct $i, j$.

As a special case, if $n = 3$ then it is called by $D(n)$ - Diophantine triple.

**Definition 1.2.** ([8]) If $D(n)$- triple $\{u, v, w\}$ satisfies the following condition

$$ (w - v - u)^2 = 4(u, v + n) \quad (1.1) $$

then $\{u, v, w\}$ is called Regular Diophantine Triple.

**Definition 1.3.** ([13,15]) Let $q$ be an odd prime and $u$ be an integer such that $gcd(u, q) = 1$. The quadratic residue symbol $(\frac{u}{q})$ is defined to be 1 or -1 according as the congruence $x^2 \equiv u \pmod{q}$ is solvable or not.

Also, Quadratic Reciprocity law was formulated by Euler although Legendre discovered it independently of Euler in 1785. We can see some results on this law as follows:

**Theorem 1.1.** ([12]) (Quadratic Reciprocity Law) Let $p, q$ be different odd primes. Then,

(i) If $p \equiv 1 \pmod{4}$ or $q \equiv 1 \pmod{4}$, then $p$ is a square (mod $q$) if and only if $q$ is a square (mod $p$).

(ii) If $p \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{4}$, then $p$ is a square (mod $q$) if and only if $q$ is not a square (mod $p$).

**Theorem 1.2.** ([2,12,14,22]) (First Supplement to the Quadratic Reciprocity Law) Let $q$ be an odd prime. Then, -1 is a square (mod $q$) necessary and sufficient condition $q \equiv 1 \pmod{4}$ holds.
Theorem 1.3. ([12,14]) (Second Supplement to the Quadratic Reciprocity Law) Let \( p \) be an odd prime. Then \( 2 \) is a square mod \( q \) if and only if \( q \equiv 1, 7 \pmod{4} \).

Definition 1.4. ([5,7]) The symbol \( \left( \frac{a}{q} \right) \) is called Legendre Symbol, if \( \left( \frac{a}{q} \right) \) equals to 1. It means that \( a \) is a quadratic residue.

\[
\left( \frac{a}{q} \right) = \begin{cases} 
1 & \text{if } a \text{ is a quadratic residue modulo } q \\
-1 & \text{if } a \text{ is a non-quadratic residue modulo } q
\end{cases}
\] (1.2)

Proposition 1.1. ([5,7]) Let \( q \) be an odd prime. Then following properties are satisfied.

(a) \( m \equiv n \pmod{q} \) implies \( \left( \frac{m}{q} \right) = \left( \frac{n}{q} \right) \)

(b) The Legendre Symbol is multiplicative: \( \left( \frac{m}{q} \right) \cdot \left( \frac{n}{q} \right) = \left( \frac{m \cdot n}{q} \right) \) where \( m, n \) are integers and coprime to \( q \) prime.

### 2. Theorems And Results

Theorem 2.1. \( P_{+31} = \{2, 9, 25\} \) is regular but non-extendible Diophantine triple.

**Proof.** For regularity, we consider the condition (1.1) of Definition 1.2. So, it is seen that \( P_{+31} = \{2, 9, 25\} \) is a regular triple. We suppose that \( \{2, 9, 25\} \) can extendible to Diophantine quadruple for any positive integer \( \vartheta \) and \( \{2, 9, 25, \vartheta\} \) is a \( P_{+31} \) quadruple. Then, there are \( a_1, a_2, a_3 \) integers such that,

\[
\begin{align*}
2 \vartheta + 31 &= a_1^2 \\
9 \vartheta + 31 &= a_2^2 \\
25 \vartheta + 31 &= a_3^2
\end{align*}
\] (2.1-2.3)

Eliminating \( \vartheta \) between (2.2) and (2.3), we obtain following equation:

\[25a_2^2 - 9a_3^2 = 496\] (2.4)

Applying factorization method on the (2.4), we get a table as follows:

| Table 2.1. Solutions of \( 25a_2^2 - 9a_3^2 = 496 \) |
|----------------|----------------|----------------|
| \((a_2, a_3)\) | 1.Class of Solutions | 2.Class of Solutions |
| \((\pm 25, \pm 41)\) | \((\pm 7, \pm 9)\) |
Considering (2.1) and (2.2), we obtain
\[ 9a_1^2 - 2a_2^2 = 217 \]  
(2.5)
Considering above solutions, we have \( a_2^2 = 625, a_2^2 = 49 \). If we put these values into the (2.5) we have \( a_1^2 = 163, a_1^2 = 35 \). This is a contradiction since \( a_1 \) isn’t an integer.

So, there is no such \( \theta \) positive integer and \( P_{+31} = \{2, 9, 25\} \) can be non-extended to \( P_{+31} \) Diophantine quadruple.

**Theorem 2.2.** A set \( P_{+31} = \{3, 46, 75\} \) is both regular and non-extendible to the \( P_{+31} \) Diophantine quadruple.

**Proof.** \( P_{+31} = \{3, 46, 75\} \) satisfies (1.1) regularity condition of Definition 1.2. So, it is regular. Supposing that \( P_{+31} = \{3, 46, 75, \omega\} \) be a \( P_{+31} \) Diophantine quadruple for positive integer \( \omega \). There are \( b_1, b_2, b_3 \) integers such that
\[ 3\omega + 31 = b_1^2 \]  
(2.6)
\[ 46\omega + 31 = b_2^2 \]  
(2.7)
\[ 75\omega + 31 = b_3^2 \]  
(2.8)
hold. Dropping \( \omega \) from (2.6) and (2.8), we have
\[ 25b_1^2 - b_3^2 = 744 \]  
(2.9)
If we use factorization method into the (2.9), we obtain solutions in a following table:

<table>
<thead>
<tr>
<th>Solutions</th>
<th>1.Class of Solutions</th>
<th>2.Class of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>((b_1, b_3))</td>
<td>(\mp19, \mp91)</td>
<td>(\mp13, \mp59)</td>
</tr>
</tbody>
</table>

Dropping \( \omega \) from (2.6) and (2.7), then we have
\[ -3b_2^2 + 46b_1^2 = 1333 \]  
(2.10)
From Table 2.2, we have \( b_1^2 = 361 \) or \( b_1^2 = 169 \). If we substitute them into the (2.10), we obtain \( b_2^2 = 5091 \) or \( b_2^2 = 2147 \) respectively. It is a contradiction and \( b_2 \) isn’t integer solution for (2.10).

Thus, there is not positive integer \( \omega \) and \( P_{+31} = \{3, 46, 75\} \) can not be extended to \( P_{+31} \) Diophantine quadruple.
Theorem 2.3. $P_{+31} = \{3,75,110\}$ is a regular triple but can not nonextendible to $P_{+31}$ Diophantine triple.

Proof. First of all, let’s show that $P_{+31} = \{3,75,110\}$ is a regular triple. If we use regularity condition for $P_{+31} = \{3,75,110\}$, it is easily seen that the set holds condition (1.1). That is why, set is regular.

Similarly, let us assume that $P_{+31} = \{3,75,110, \alpha\}$ be a Diophantine quadruple for positive integer $\alpha$. So, we get $c_1, c_2, c_3 \in \mathbb{Z}$ such that following equations are satisfied.

$$3 \alpha + 31 = c_1^2 \quad (2.11)$$
$$75 \alpha + 31 = c_2^2 \quad (2.12)$$
$$110 \alpha + 31 = c_3^2 \quad (2.13)$$

If we reduce $\alpha$ from (2.11) and (2.12), we obtain an equation as same as (2.9) for $(c_1, c_2)$. So, we have the solutions as same as Table 2.2 for $(c_1, c_2)$. Dropping $\alpha$ from (2.11) and (2.13), then we obtain

$$110c_1^2 - 3c_3^2 = 3317 \quad (2.14)$$

Substituting $c_1^2 = 361$ or $c_1^2 = 169$ into the (2.14), $c_3^2 = 12131$, or $c_3^2 = 5091$ are got. It is seen that $c_3$ is not integer solution for (2.14). Thus, it is a contradiction.

Therefore, there is no positive integer $\alpha$ and also $P_{+31} = \{3,75,110\}$ can not be extended to $P_{+31}$ Diophantine quadruple.

Theorem 2.4. A set $P_{+31} = \{9,25,66\}$ regular diophantine triple and nonextendible to $P_{+31}$ quadruple.

Proof. $P_{+31} = \{9,25,66\}$ holds (1.1) condition in the Definition 1.2. That is why it is regular. Assume that $P_{+31} = \{9,25,66, \varrho\}$ is Diophantine quadruple for $\varrho \in \mathbb{Z}^+$. Definition 1.1 implies that

$$9 \varrho + 31 = d_1^2 \quad (2.15)$$
$$25 \varrho + 31 = d_2^2 \quad (2.16)$$
$$66 \varrho + 31 = d_3^2 \quad (2.17)$$

for $d_1, d_2, d_3 \in \mathbb{Z}$. Simplification of (2.15) and (2.16), we have;

$$25d_1^2 - 9d_2^2 = 496 \quad (2.18)$$

This equation is similar to (2.4) for $(d_1, d_2)$. From Table 2.1, we obtain $d_1^2 = 625$, $d_2^2 = 49$. From (2.15) and (2.17), we obtain
\[ 22d_1^2 - 3d_3^2 = 589 \]  
(2.19)

Substituting \( d_1^2 = 625 \), \( d_1^2 = 49 \) into the (2.19), we have \( d_3^2 = 4387 \) and \( d_3^2 = 163 \), respectively. This is a contradiction since \( d_3 \) is not integer solution of (2.19).

Hence, \( P_{+31} = \{9,25,66\} \) can not extendible to \( P_{+31} \) Diophantine quadruple.

**Theorem 2.5.** Both \( P_{-31} = \{2,100,128\} \) and \( P_{-31} = \{2,128,160\} \) are regular Diophantine triple and also non-extendible.

**Proof.** We can see that both \( P_{-31} = \{2,100,128\} \) and \( P_{-31} = \{2,128,160\} \) are regular Diophantine triples from (1.1) condition.

Suggesting that \( \{2,100,128,h\} \) is a \( P_{-31} \) Diophantine quadruple for positive integer \( h \). Then, there are \( e_1, e_2, e_3 \in \mathbb{Z} \) such that

\[ 2h - 31 = e_1^2 \]  
(2.20)

\[ 100h - 31 = e_2^2 \]  
(2.21)

\[ 128h - 31 = e_3^2 \]  
(2.22)

Simplifying \( h \) between (2.20) and (2.22), we get

\[ -64e_1^2 + e_3^2 = 1953 \]  
(2.23)

and similarly from (2.20) and (2.21)

\[ -50e_1^2 + e_2^2 = 1519 \]  
(2.24)

By factorizing (2.23), we obtain following table for solutions.

**Table 2.3.** Solutions of \( -64e_1^2 + e_3^2 = 1953 \)

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>( (e_3, e_1) )</td>
<td>( (+977, +122) )</td>
<td>( (+143, +17) )</td>
<td>( (+113, +13) )</td>
<td>( (+47, +2) )</td>
</tr>
</tbody>
</table>

By substituting \( e_1^2 = 14884 \), \( e_1^2 = 289 \), \( e_1^2 = 169 \), \( e_1^2 = 4 \) into the (2.24), we get \( e_2^2 = 745719 \), \( e_2^2 = 15969 \), \( e_2^2 = 8450 \), \( e_2^2 = 1719 \) respectively. It is seen that it is a contradiction since \( e_2 \notin \mathbb{Z} \).

As a consequence, \( P_{-31} = \{2,100,128\} \) can not be extended to \( P_{-31} \) Diophantine quadruple.
Let \( P_{-31} = \{2, 128, 160, \mathcal{L}\} \) be a Diophantine quadruple for \( \mathcal{L} \in \mathbb{Z}^+ \). From Definition 1.1, we have

\[
\begin{align*}
2\mathcal{L} - 31 &= f_1^2 \\
128\mathcal{L} - 31 &= f_2^2 \\
160\mathcal{L} - 31 &= f_3^2
\end{align*}
\]

for \( f_1, f_2, f_3 \in \mathbb{Z} \). Dropping \( \mathcal{L} \) from (2.25) and (2.26), we have an equation like (2.19).

Hence, Table 2.3 can be used for \((f_2, f_1)\) instead of \((e_3, e_1)\). From (2.25) and (2.27), we also have

\[-80f_1^2 + f_3^2 = 2449 \tag{2.28}\]

Putting \( f_1^2 = 14884, \ f_1^2 = 289, \ f_1^2 = 169, \ f_1^2 = 4 \) into the (2.28), we have \( f_3^2 = 1193169, \ f_3^2 = 25569, \ f_3^2 = 15969, \ f_3^2 = 2749 \). It is a contradiction because \( f_3 \) is not an integer solution of (2.28).

Therefore, \( P_{-31} = \{2, 128, 160\} \) is nonextendable to \( P_{-31} \) Diophantine triple.

**Theorem 2.6.** A set \( P_{-31} = \{4, 64, 98\} \) is not only regular but also nonextendible Diophantine triple.

**Proof.** Regularity of \( P_{-31} = \{4, 64, 98\} \) can be easily seen from (1.1). In the same way, supposing that \( P_{-31} = \{4, 64, 98, \mathcal{M}\} \) be a Diophantine quadruple for positive integer \( \mathcal{M} \). We get \( g_1, g_2, g_3 \in \mathbb{Z} \) such that

\[
\begin{align*}
4\mathcal{M} - 31 &= g_1^2 \\
64\mathcal{M} - 31 &= g_2^2 \\
98\mathcal{M} - 31 &= g_3^2
\end{align*}
\]  

Dropping \( \mathcal{M} \) from (2.29) and (2.30), we get an equation as follows:

\[g_2^2 - 16g_1^2 = 465 \tag{2.32}\]

and if we eliminate \( \mathcal{M} \) from (2.29) and (2.31), then

\[2g_3^2 - 49g_1^2 = 1457 \tag{2.33}\]

is obtained. Table 2.4 is got from (2.32) as follows:
Table 2.4. Solutions of \( g_2^2 - 16g_1^2 = 465 \)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>((g_2, g_1))</td>
<td>((\mp 233, \mp 58))</td>
<td>((\mp 79, \mp 19))</td>
<td>((\mp 49, \mp 11))</td>
<td>((\mp 23, \mp 2))</td>
</tr>
</tbody>
</table>

Substituting \( g_1^2 = 3364, g_1^2 = 361, g_1^2 = 121, g_1^2 = 4, \) into the (2.33), \( g_3^2 = \frac{166293}{2}, \) \( g_3^2 = 9573, g_3^2 = 3693, g_3^2 = \frac{1653}{2} \) are got. it is a contradiction since \( g_3 \) is not integer solution for (2.33)

So, \( P_{-31} = \{4, 64, 98\} \) can not be extended to \( P_{-31} \) Diophantine quadruple.

**Theorem 2.7.** Following conditions are satisfied for \( P_{\pm 31} \) sets.

(a) There isn’t any set \( P_{+31} \) includes any multiplication of 4, 7, 13, or 19.

(b) There is no set \( P_{-31} \) involves any multiplication of 3, 11, 13, or 17.

**Proof.** (a) (i) Supposing that \( a \) is an element of set \( P_{+31} \). If \( 4m \in P_{+31} \) for \( m \in \mathbb{Z} \), then

\[
4ma + 31 = X^2
\]  

(2.34)
satisfy for some integer \( X \). Applying \((\text{mod } 4)\) on the (2.34), we obtain

\[
X^2 \equiv 3 \pmod{4}
\]  

(2.35)
Since \( X \in \mathbb{Z} \), then \( X \) is even or odd integer. So, (2.35) can not has a solution. This is a contradiction. Hence, \( 4m \notin P_{+31} \) for \( m \in \mathbb{Z} \).

(ii) Similarly, assuming that \( b \in P_{+31} \) and \( 7n \in P_{+31} \) for \((n \in \mathbb{Z})\). So,

\[
7nb + 31 = \Psi^2
\]  

(2.36)
holds for integer \( \Psi \). By \((\text{mod } 7)\), we obtain

\[
\Psi^2 \equiv 3 \pmod{7}
\]  

(2.37)
From Theorem 1.1 and Definition 1.4,

\[
\left(\frac{3}{7}\right)\left(\frac{7}{3}\right) = (-1)^{\frac{3-1}{2} \cdot \frac{7-1}{2}} = -1
\]  

(2.38)
is hold. Using Proposition 1.1, we have \( \left(\frac{7}{3}\right) = \left(\frac{1}{3}\right) = +1 \) and substituting it into the (2.38) then \( \left(\frac{3}{7}\right) = -1 \) is obtained. This implies that equivalent (2.38) isn’t solvable. So, \( 7n \notin P_{+31} \) for \( n \in \mathbb{Z} \).
(iii) In the same vein, if \( c \in P_{+31} \) and \( 13k \in P_{+31} \) for \( k \in \mathbb{Z} \), then
\[ 13kc + 31 = \Omega^2 \] (2.39)
holds for integer \( \Omega \). Applying \((\text{mod } 13)\) on (2.39), we have
\[ \Omega^2 \equiv 5 \ (\text{mod } 13) \] (2.40)
Using Theorem 1.1 and Definition 1.4, then
\[ \left( \frac{5}{13} \right) \left( \frac{13}{5} \right) = (-1)^{\frac{5-1}{2} \cdot \frac{13-1}{2}} = +1 \] (2.41)
is satisfied. By substituting \( \left( \frac{13}{5} \right) = \left( \frac{3}{5} \right) = -1 \) into the (2.41) then we get \( \left( \frac{5}{13} \right) = -1 \). This is a contradiction since (2.40) can not solvable. Therefore, \( 13k \notin P_{+31} \) for \( k \in \mathbb{Z} \).

(iv) Assume that if \( d \in P_{+31} \) and \( 19l \in P_{+31} \). Then,
\[ 19ld + 31 = T^2 \] (2.39)
holds for integer \( T \). Applying \((\text{mod } 19)\) on (2.39), we get
\[ T^2 \equiv 12 \ (\text{mod } 19) \] (2.40)
From Legendre symbol’s properties and Theorem 1.3, we have
\[ \left( \frac{12}{19} \right) = \left( \frac{4}{19} \right) \cdot \left( \frac{3}{19} \right) = \left( \frac{2}{19} \right) \left( \frac{2}{19} \right) \left( \frac{3}{19} \right) \] (2.41)
Using Definition 1.4, we obtain \( \left( \frac{3}{19} \right) = -1 \). It requires that \( \left( \frac{12}{19} \right) = -1 \) and it is a contradiction So, \( 19l \notin P_{+31} \) for \( l \in \mathbb{Z} \).

(b) Along the same line, assume that any multiply of 3,11, 13, or 17 are in the \( P_{-31} \).

Briefly and respectively, we get
\[ A^2 \equiv 2 \ (\text{mod } 3) \] (2.42)
\[ B^2 \equiv 2 \ (\text{mod } 11) \] (2.43)
\[ C^2 \equiv 8 \ (\text{mod } 13) \] (2.44)
\[ D^2 \equiv 3 \ (\text{mod } 17) \] (2.45)

From Theorem 1.3, we calculate \( \left( \frac{2}{3} \right) = -1 \), \( \left( \frac{2}{11} \right) = -1 \) and \( \left( \frac{2}{13} \right) = -1 \). This is a contradiction since (2.42), (2.43) and (2.44) are unsolvable. Besides, from Theorem 1.1 and Definition 1.4, we get \( \left( \frac{3}{17} \right) = -1 \) which means that (2.45) is not solvable. So, any multiplication of 3,11, 13, or 17 are not in the \( P_{-31} \).
Remark 2.8. One may find different regular non-extendible triples $P_{+31}$ or $P_{-31}$ and also extend the Theorem 2.7 using our method.

References


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