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CONVERGENCE OF THE PICARD-MANN HYBRID ITERATION IN CONVEX CONE METRIC SPACES

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Abstract : In this study we try to show convergence of the Picard-Mann hybrid Iteration in convex cone metric spaces for common fixed points of infinite families of uniformly quasi-Lipschitzian mappings and quasi-nonexpansive mappings. A convex cone mtric space is a cone metric space with a convex structure.

Keywords convex metric space, convex structure, convex cone metric spaces, Picard-Mann hybrid iteration

1. Introduction and Preliminaries

Takashi [1] introduced the concept of convex metric space which is a more general space and every linear normed space is a special case of a convex metric space. In 2005, Tian [2] gave some important and necessary rules such that the Ishikawa iteration sequence for an asymptotically quasi-nonexpansive mapping to converge to a fixed point in convex metric spaces. In 2009, Wang and Liu [3] had some vital and needful rules rules for an Ishikawa iteration sequence with errors to approximate a common fixed point of two uniformly quasi-Lipschitzian mappings in convex metric spaces. Finally Chang et al. [4] and Liu [5] et al. had some significant and necessary conditions for Ishikawa iteration process with errors to approximate common fixed points of infinite families of uniformly quasi-Lipschitzian mappings in convex metric spaces. Furthermore many authors shown that some extra structures on convex metric spaces like "cone". It is desired to put the convex structure on cone metric spaces [6], [7], [1], [5]. and consider Picard-Mann hybrid Iteration scheme [8] with errors to approximate a common fixed points of two infinite families of uniformly quasi-Lipschitzian mappings in convex cone metric spaces.

Throught this paper, E is a normed vector space with a normal solid cone P.

Definition 1.1. [9] A nonempty subset P of E is called a cone if P is closed, $P \neq \{\theta\}$, for $a, b \in \mathbb{R}^+ = [0, \infty)$ and $x, y \in P$, $ax + by \in P$ and $P \cap \{-P\} = \{\theta\}$. We define a partial ordering $\preccurlyeq$ in E as $x \preccurlyeq y$ if $y - x \in P$. $x \ll y$ shows that $y - x \in intP$ and $x \prec y$ means that $x \preccurlyeq y$ but $x \neq y$. A cone P is said to be solid if its interior $intP$ is nonempty. A cone P is said to be normal if there exists a positive number k such that for $x, y \in P$, $0 \preccurlyeq x \preccurlyeq y$ implies $\|x\| \leq k\|y\|$. The least positive number k is called the normal constant of P.

Definition 1.2. [9] Let X be a nonempty set. A mapping $d: X \times X \to (E, P)$ is called a cone metric if ;

(i) for $x, y \in X$, $0 \preccurlyeq d(x, y)$ and $d(x, y) = \theta$ iff $x = y$,
(ii) for $x, y \in X$, $d(x, y) = d(y, x)$,

(iii) for $x, y, z \in X$, $d(x, y) \preccurlyeq d(x, z) + d(z, y)$.
A nonempty set X with a cone metric $d: X \times X \to (E, P)$ is called a cone metric space denoted by (X, d),where P is a solid normal cone.

Definition 1.3. [5] Let (X, d) be a cone metric space and $T: (X, d) \to (X, d)$ a mapping.

(i) T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ with $d(T^n x, T^n y) \preccurlyeq k_n d(x, y)$ for all $x, y \in X$ $(n \in \mathbb{N} \cup \{0\})$.
(ii) T is said to be asymptotically quasi-nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ with $d(T^n x, p) \preccurlyeq k_n d(x, p)$ for all $x \in X$ $p \in F(T)$ the set of fixed points of T $(n \in \mathbb{N} \cup \{0\})$.
(iii) T is said to be uniformly quasi-Lipschitzian if there exists a sequence $L > 0$ such that $d(T^n x, p) \preccurlyeq Ld(x, p)$ for all $x \in X$ $p \in F(T)$ the set of fixed points of T $(n \in \mathbb{N} \cup \{0\})$.

Remark 1.1. If $F(T) \neq \emptyset$, then (i) implies (ii) implies, and (ii) implies (iii) implies by putting $L = sup_{n\geq 0} k_n < \infty$, but not conversely from some examples shown in metric spaces [**5**].

Now we think about that the convex structure in cone metric spaces.

Definition 1.4. Let (X, d) be a cone metric space. A mapping $W: X^3 \times I^3 \to X$ is called a convex structure on X if $d(W(x, y, z, a_n, b_n, c_n), u) \preccurlyeq a_n d(x, u) + b_n d(y, u) + c_n d(z, u)$ for real sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $I = [0,1]$ satisfying $a_n + b_n + c_n = 1$ $(n \in \mathbb{N})$ and x, y, z and $u \in X$. A cone metric space (X, d) with a convex structure W is called a convex cone metric space and shown as (X, d, W). A nonempty subset C of a convex cone metric space (X, d, W) si said to be convex if $W(x, y, z, a, b, c) \in C$ for all $x, y, z \in C$ and $a, b, c \in I$.

Definition 1.5. Let (X, d) be a cone metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for any $c \in E$ with $0 \ll c$, there exists s natural number N such that for all $n > N$, $d(x_n, x) \ll c$, then , $\{x_n\}$ is said to be converge to x and denoted as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ (as $n \to \infty$).

(ii) If for any $c \in E$ with $0 \ll c$, there exists natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then , $\{x_n\}$ is called a Cauchy sequence in X.

(iii) (X, d) is said to be complete if every Cauchy sequence converges.

Proposition 1.1 [9] Let $\{x_n\}$ be asequence in a cone metric space (X, d) and P be a normal cone. Then

(i) $\{x_n\}$ converges to x in X if and only if $d(x_n, x) \to 0$ (as $n \to \infty$) in E.

(ii) $\{x_n\}$ is a Cauchy sequence in X if and only if $d(x_n, x_m) \to 0$ (as $n, m \to \infty$) in E.

Suppose that C be a nonempty convex subset of a convex cone metric space (X, d, W). Let $S_i: C \to C$ be quasi-nonexpansive mappings, $T_i: C \to C$ be uniformly quasi-Lipschitzian mappings with Lipschitzian constants $L_i > 0$, $(i \in \mathbb{N})$.

Let $\{\alpha_n\}$ be a sequence in C and in $[0,1]$. For any given $x_0 \in C$, $\{x_n\}$ defined as

$$\begin{cases} x_{n+1} = S_n y_n \\ y_{n} = W(x_n, T_n^n x_n, \alpha_n) \end{cases}$$

is called the Picard-Mann hybrid Iteration process with errors for two sequences $\{S_i\}_{i\in\mathbb{N}}$ and $\{T_i\}_{i\in\mathbb{N}}$.

In this study, we present convergence theorem concerning Picard-Mann hybrid Iteration process with errors for approximating a common fixed point of two sequences quasi-nonexpansive mappings, uniformly quasi-Lipschitzian mappings in convex cone metric spaces

2. Main Results

Lemma 2.1. [7] Let $\{p_n\}$, $\{q_n\}$ and $\{r_n\}$ be sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} q_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$. If $p_{n+1} \leq (1+q_n)p_n + r_n$, $(n \in \mathbb{N} \cup \{0\})$ then

(i) $\lim_{n\to\infty} p_n$ exists, and

(ii) If $\underline{\lim}_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Theorem 2.2. Let $d: X \times X \to (E, P)$ be a cone metric, where P s a solid normal cone with the normal constant k. Let C be a nonempty closed convex subset of complete convex metric space (X, d, W) and $S_i: C \to C$ be quasi-nonexpansive mappings, $T_i: C \to C$ be uniformly quasi-Lipschitzian mappings with Lipschitzian constants $L_i > 0$, $(i \in \mathbb{N})$. Suppose that $\mathcal{F} := (\bigcap_{i=1}^{\infty} F(S_i)) \cap (\bigcap_{i=1}^{\infty} F(T_i))$ is nonempty and bounded. Let $\{x_n\}$ be a sequence in C defined by (1.1) with bounded sequence $\{\alpha_n\}$ in C and in $[0,1]$. Assume that L_i is bounded.

Then the followings are equivalent;

(i) $\{x_n\}$ converges to a common fixed point of $p \in \mathcal{F}$,
(ii) $\underline{\lim}_{n\to\infty} d(x_n, \mathcal{F}) = 0$, where $d(x, \mathcal{F}) = 0 = inf d(x,q)_{q\in\mathcal{F}}$

Proof. Clearly, (i) implies (ii). Now we think that (ii) implies (i).

(a) First of all, we show that $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$. For any $p \in \mathcal{F}$, from (1.1) we get

$$d(x_{n+1}, p) \preccurlyeq d(S_n y_n, p) \preccurlyeq d(y_n, p)$$

and

$$d(y_n, p) = d(W(x_n, T_n^n x_n, \alpha_n), p)$$

$$d(y_n, p) \preccurlyeq (1-\alpha_n) d(x_n, p) + \alpha_n d(T_n^n x_n, p)$$

$$d(y_n, p) \preccurlyeq (1-\alpha_n) d(x_n, p) + \alpha_n L d(x_n, p)$$

$$d(y_n, p) \preccurlyeq (1-\alpha_n + \alpha_n L) d(x_n, p)$$

So, we have

$$d(x_{n+1}, p) \preccurlyeq (1-\alpha_n + \alpha_n L) d(x_n, p)$$

$$d(x_{n+1}, p) \preccurlyeq (1 + \alpha_n(1 - L))d(x_n, p)$$

Thus, by the normality of P, for the normal constant $k > 0$.

$$\|d(x_{n+1}, p)\| \leq k(1 + \alpha_n(1 - L))\|d(x_n, p)\|$$

As p is an arbitrary point in $\mathcal{F}$,

$$\|d(x_{n+1}, \mathcal{F})\| \leq k(1 + \alpha_n(1 - L))\|d(x_n, \mathcal{F})\|$$

Since $\sum_{n=1}^{\infty} \alpha_n < \infty$ by Lemma 2.1. $\lim_{n\to\infty} \|d(x_n, \mathcal{F})\|$ exists, so $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists. Now $\underline{\lim}_{n\to\infty} d(x_n, \mathcal{F}) = 0$ implies $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

(b) Secondly, we have that $\{x_n\}$ is a Cauchy sequence in C.

Since $\lim_{n\to\infty} \|d(x_n, \mathcal{F})\| = 0$ and $\sum_{n=1}^{\infty} \alpha_n < \infty$ for any positive real number ε, there exists a natural number $N_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \|d(x_n, \mathcal{F})\| \leq \frac{\varepsilon}{M_1+1}$ for $n \geq N_0$, where $M_1 = e^{(L-1)\sum_{k=0}^{\infty} \alpha_k}$.

Especially, there exist a point $p_1 \in \mathcal{F}$ and a positive integer $N_1 > N_0$ such that

$$\left\|d(x_{N_1}, p_1)\right\| \leq \frac{\varepsilon}{M_1 + 1}$$

On the other side, by the fact that

$$\|d(x_{n+1}, p)\| \leq k(1 + \alpha_n(1 - L))\|d(x_n, p)\|$$

and the inequality $1 + x \leq e^x$ for $x \geq 0$, we get

$$\begin{aligned}
\|d(x_{n+m}, p)\| &\leq \left(1 + \alpha_{n+m-1}(L - 1)\right)\|d(x_{n+m-1}, p)\| \\
&\leq e^{\alpha_{n+m-1(L-1)}}\|d(x_{n+m-1}, p)\| \\
&\leq e^{\alpha_{n+m-1(L-1)}}(1 + \alpha_{n+m-2}(L - 1)\|d(x_{n+m-2}, p)\| \\
&\leq e^{\alpha_{n+m-1+n+m-2(L-1)}}\|d(x_{n+m-2}, p)\| \\
&\ \vdots \\
&\leq M_1\|d(x_n, p)\|
\end{aligned}$$

Thus we have for $m + n, n > N_1$,

$$\begin{aligned}
\|d(x_{n+m}, x_n)\| &\leq \|d(x_{n+m}, p_1)\| + \|d(x_n, p_1)\| \\
&\leq M_1\left\|d(x_{N_1}, p_1)\right\| + \left\|d(x_{N_1}, p_1)\right\| \\
&\leq (1 + M_1)\left\|d(x_{N_1}, p_1)\right\| \\
&\leq (1 + M_1)\frac{\varepsilon}{M_1 + 1}
\end{aligned}$$

On the other side, for $c \in E$ with $0 \ll c$ there exists a positive number δ such that for $d \in E$ with $\|d\| < \delta$, $c - d \in intP$. From the fact that $d(x_n, x_m) \to 0$ (as $n, m \to \infty$) in E, for such δ, there exists a natural number N such that for all $n, m \geq N$ $\| d(x_n, x_m)\| < \delta$. Thus $c - d\ d(x_n, x_m) \in intP$, which means that $\{x_n\}$ is a Cauchy sequence in C by Proposition1.1. Since X is complete and C is closed, $\{x_n\}$ converges to some point p^* in C.

(c) Finally, we show that $p^* \in \mathcal{F}$. Let $\{p_n\}$ be sequence in $\mathcal{F}$ such that $p_n \to p'$. Since

$$\begin{aligned} d(p', T_i p') &\preccurlyeq d(p', p_n) + d(p_n, T_i p') \\ d(p', T_i p') &= d(p', p_n) + d(T_i p_n, T_i p') \\ &\preccurlyeq d(p', p_n) + Ld(p_n, p') \\ &\to 0 \text{ (as } n \to \infty), \end{aligned}$$

$d(p', T_i p') = 0$ for $i \in \mathbb{N}$. Similarly $d(p', S_i p') = 0$ for $i \in \mathbb{N}$.

Thus $p' \in \mathcal{F}$, means that $\mathcal{F}$ is closed. Since $d(p^*, \mathcal{F}) = 0 = \lim_{n\to\infty} d(x_n, \mathcal{F})$ we get $p^* \in \mathcal{F}$, which completes proof.

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