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Some P_s Diophantine Triples for Especial s Integer

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Abstract

The aim of this research paper is to consider some P_s –Diophantine triples including integer numbers under the special condition for $s = 55$. It is demonstrated that they can not be extended to P_{55} –Diophantine quadruples but they are regular. Also, several properties on the elements of this type sets are obtained. Some notations such as Modular Arithmetic, Quadratic Reciprocity or Residue Law, Legendre Symbol are used to prove our results.

Keywords: Diophantine m-Tuples, P_s Diophantine Triple, Quadratic Residue and Reciprocity Law, Legendre Symbol, Prime Factorization.

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Introduction and Preliminaries

Many problems have been remained to solve on the topic Diophantine triples over the set of integers, even if it is a very old topic. An aim of the works is to appear structures in the number theory. That is why, we try to obtain some result on the structures of the such sets.

Our aim is to obtain some results on the characterization and classification of the $\boldsymbol{P_s}$ −Diophantine triples for $\boldsymbol{s = 55}$. We prove that several $\boldsymbol{P_{55}}$ −Diophantine triples can not be extended to $\boldsymbol{P_{55}}$ −Diophantine triples but they are regular. Also, we demonstrated that some of the integers are not in the such $\boldsymbol{P_{55}}$ − Diophantine sets.

Dujella [2-3] explained and obtained important results on the Diophantine sets. Özer [8-14] determined some new results using different techniques for Diophantine triples with property $\boldsymbol{P_s}$. Gopalan et al [4-5,17] solved both general type and special types of the Diophantine and Pell equations and defined strong outcomes. The books of authors Cohen [1], Mollin [7], Silverman [16] and [6,15-16] references are used for many number theorists as well as us since include notions of the number theory very well.

We would like to give the general definition of the such sets and some important concepts as follows:

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Definition 1.1.

Let a set $H = \{h_1, h_2, h, \ldots, h_m\}$ includes m different positive integers. H is called Diophantine m -tuple with the property P_s over the set of integers if $h_i h_j + s$ is a perfect square integer where $i, j = 1, \ldots, r$ and i, j are different from each others.

In the above definition, we can say that it is P_s −Diophantine triple if m=3.

Definition 1.2.

It is called that a is a quadratic residue $(mod\ t)$ if there exist an integer x such that $x^2 \equiv a\ (mod\ t)$ for $t \in \mathbb{N}$, $a \in \mathbb{Z}$ and the equivalent has at least one solution.

Definition 1.3.

Let $a \in \mathbb{Z}$ be integer and $p > 2$ be a prime number. Then, $\left(\frac{a}{p}\right)$ is called the Legendre Symbol and determined as follows.

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is quadratic residue modulo } p \\ -1 & \text{if } a \text{ is quadratic nonresidue modulo } p \\ 0 & \text{if } p \mid a \end{cases}$$

Definition 1.4.

If $\boldsymbol{D(s)}$- Diophantine triple $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ satisfies the condition

$$(\boldsymbol{\gamma} - \boldsymbol{\beta} - \boldsymbol{\alpha})^2 = 4(\boldsymbol{\alpha}.\boldsymbol{\beta} + \boldsymbol{s}) \qquad (\mathbf{1.1})$$

Then, it is called $D(s)$- Regular Diophantine Triple.

Theorem 1.1.

Let R, T be distinct odd primes. Then

$$\left(\frac{R}{T}\right)\left(\frac{T}{R}\right) = (-1)^{\frac{R-1}{2} \cdot \frac{T-1}{2}}$$

where $\left(\frac{\cdot}{\cdot}\right)$ represents Legendre symbol. This equation is called as The Reciprocity Law.

2. Main Results and Conclusions

Theorem 2.1. $P_{+55} = \{1, 9, 26\}$ regular P_{+55} −Diophantine triple but can not extended to P_{+55} −Diophantine quadruple.

Proof. First of all, we can consider regularity condition defined in Definition 1.4. If we apply (1.1) it is easily seen that $P_{+29} = \{1, 167, 196\}$ is a regular triple. Suppose that $\{1, 9, 26\}$ can be extended such as $\{1, 167, 196, a\}$ is a P_{+55} −Diophantine quadruple for $a \in \mathbb{N}$. So, we can say that there are X_1, X_2, X_3 integers such that;

$$a + 55 = {X_1}^2 \qquad (1)$$

$$9a + 55 = {X_2}^2 \qquad (2)$$

$$26a + 55 = {X_3}^2 \qquad (3)$$

Removing a between (1) and (2), we have

$$9{X_1}^2 - {X_2}^2 = 440 \qquad (4)$$

Similarly,removing a between (1) and (3), also (2) and (3), following equatipons are hold.

$$26X_1{}^2 - X_3{}^2 = 1375 \tag{5}$$

$$29X_2{}^2 - 93^2 = 385 \tag{6}$$

If we consider (4), (5) and (6), the number of integer solutions for (6) is zero (-0-). This implies that we can not solve the system (4)-(5)-(6) in the set of integers. This is a contradiction.

So, $P_{+55} = \{1, 9, 26\}$ can not be extended P_{+55} −Diophantine quadruple.

Theorem 2.2. $P_{+55} = \{\ 9, 34, 81\}$ is a regular non-extendible P_{+55} −Diophantine triple.

Proof. Using preliminaries section and considering (1.1), it is trivial that $P_{+55} = \{\ 9, 34, 81\}$ is a regular triple. Assuming that $\{\ 9, 34, 81\}$ is extended to dıophantine quadruple like $\{9, 34, 81, b\}$ for $b \in \mathbb{N}$. Then, there are Y_1, Y_2, Y_3 integers satisfying following equations:

$$9b + 55 = Y_1{}^2 \tag{7}$$

$$34b + 55 = Y_2{}^2 \tag{8}$$

$$81b + 55 = Y_3{}^2 \tag{9}$$

Dropping b between (7) and (9), we obtain

$$9Y_1{}^2 - Y_3{}^2 = 440 \tag{10}$$

Also, removing b between (7) - (8) as well as (8) - (9), we get

$$34Y_1{}^2 - 9Y_2{}^2 = 1375 \tag{11}$$

$$81Y_2{}^2 - 34Y_3{}^2 = 2585 \tag{12}$$

We can search integer solutions of the (10), (11), (12).

Some solutions for (10) can be seen as follows:

$$(Y_1, Y_3) = \ (\mp 37, \mp 109), (\mp 19, \mp 53), (\mp 9, \mp 17), (\mp 7, \mp 1)\ \ldots$$

Some solutions for (11) and (12):

$$(Y_1, Y_2) = (\mp 1934, \mp 3759), (\mp 1792, \mp 3483), (\mp 620, \mp 1205), (\mp 80, \mp 155),$$
$$(\mp 28, \mp 53), (\mp 26, \mp 49), (\mp 10, \mp 15),\ \ldots$$

and

$$(Y_2, Y_3) = (\mp 87, \mp 134), (\mp 19, \mp 28), \ldots$$

It is seen that this system don't have common solution in the set of integers. This is a contradiction.

So, $P_{+55} = \{9, 34, 81\}$ is a regular non-extendible P_{+55} −Diophantine triple.

Theorem 2.3. $P_{+55} = \{9, 81, 146\}$ is a regular P_{+55} −Diophantine triple but non-extendible P_{+55} −Diophantine quadruple.

Proof. We can see that $P_{+55} = \{9, 81, 146\}$ is a regular P_{+55} −Diophantine triple by applying (1.1) in the Definition 1.4. Assume that $\{9, 81, 146, c\}$ is P_{+55} −Diophantine quadruple for $c \in \mathbb{N}$. Thus, $Z_1, Z_2, Z_3 \epsilon \mathbb{Z}$ such that

$$9c + 55 = {Z_1}^2 \tag{13}$$

$$81c + 55 = {Z_2}^2 \tag{14}$$

$$146c + 55 = {Z_3}^2 \tag{15}$$

Removing c between (13) and (14), we obtain

$$9{Z_1}^2 - {Z_2}^2 = 440 \tag{16}$$

and removing c between (13) - (15), we have

$$146{Z_1}^2 - 9{Z_3}^2 = 551 \tag{17}$$

If we search integer solutions of the (17), we can not get any solutions in the set of integers. This implies that there is no integer solution of the system (13)-(14)-(15). It is a contradiction.

So, $P_{+55} = \{9, 81, 146\}$ can not be extended to P_{+55} −Diophantine quadruple.

Theorem 2.4. There is no integer divided by 4 in the P_{55}-Diophantine sets.

Proof. Let us suppose that a and $4t$ are integers in the P_{55}-Diophantine sets. $t \in \mathbb{Z}$. Then, following equation can be written;

$$4ta + 55 = \mathfrak{A}^2$$

such that it has solutions for some $\mathfrak{A}$ integers. By applying (modulo 4) we get,

$$\mathfrak{A}^2 \equiv 3(mod\ 4)$$

$\mathfrak{A}^2 \equiv 3(mod\ 4)$ has solution if Legendre symbol satisfies $\left(\frac{3}{4}\right) = 1$ or we can consider modulo 4 resıdue set. If we put integers from the modulo4 residue set, ıt is easily seen that there is no integer satisfies $\mathfrak{A}^2 \equiv 3(mod\ 4)$.

So, $\left(\frac{3}{4}\right) = -1$ and it gives a contradiction. Therefore, there is not any P_{55}-Diophantine sets contain integers multiple of 4.

Theorem 2.5. There is not any integer in the P_{55}-Diophantine sets divided by 7.

Proof. Assuming that b and $7m, (m \in \mathbb{Z})$ are integers in the P_{55}-Diophantine sets. Using Definition1.1 we obtain following equation

$$7bm + 55 = \mathfrak{B}^2$$

for integer $\mathfrak{B}$. Applying (mod 7), we obtain

$$\mathfrak{B}^2 \equiv 6\ (mod\ 7)$$

$\mathfrak{B}^2 \equiv 6\ (mod\ 7)$ can solvable if $\left(\frac{6}{7}\right) = 1$. Considering Lendre symbol properties, we have

$$\left(\frac{6}{7}\right) = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right)$$

Using Theorem 1.1 and Definition 1.3., following results are obtained.

$$\left(\frac{2}{7}\right) = +1 \quad \text{and} \quad \left(\frac{3}{7}\right) = -1$$

If we put them, we have $\left(\frac{6}{7}\right) = \left(\frac{2}{7}\right)\left(\frac{3}{7}\right)$ = -1. It implies that $\mathfrak{B}^2 \equiv 6\ (mod\ 7)$ can not have integer solutions. It occurs a contraction with hyphothesis. Thus, there is no set P_{55}-Diophantine sets contain elements multiple of 7.

Conclusion 2.1. There isn't any integers in the set P_{55}- Diophantine sets divided by 8, 12, 14, 25, 29, 31 and so on…

Proof. It is trivial. Also, Proof of the Theorem 2.5 can be done using the proofs of the Theorem 2.3. or Theorem 2.4 and notations of the Preliminaries section. One may extend Theorem 2.5 for other integers and determine general result for P_{55}-Diophantine sets.

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